# Towards a Szegö Limit Theorem for Berezin-Toeplitz Operators with Singular Symbols

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**Abstract**

This paper considers the spectra of self-adjoint Toeplitz operators for various symbols, specifically in regards to the Weyl and Berezin-Toeplitz quantizations. The main result is a proof of the Szegö Limit Theorem for a very specific type of Berezin-Toeplitz operator, from which we make a conjecture for more general symbols.

## Contents



## 1 Introduction

The spectra of self-adjoint Toeplitz operators has long been an object of study [1]. For fairly general symbols, these spectra are well understood. In fact, in [2], Ulf Grenander and Gábor Szegö proved the Szegö Limit Theorem, which describes the asymptotic distribution of eigenvalues specifically in terms of such symbols. For symbols that are generalized functions, however, the spectra of self-adjoint Toeplitz operators is not well understood. In this paper we begin by restating the findings of Grenander and Szegö, and then we extend their work

to symbols that are distributions; specifically, the Dirac delta function. We then make a conjecture that further generalizes our work. Note: MATLAB is used throughout the course of the paper to highlight various examples.

## 2 Weyl Quantization

Suppose $f(x,p) : [-\pi,\pi] \times [0,1] \to \mathbb{R}$ is a real-valued function defined on the cylinder s.t. $f(x+2\pi) = f(x)$. We can then define the Fourier series

$$f(x,p) = \sum_{k=-\infty}^{\infty} f_k(p)e^{ikx}. \tag{1}$$

On p.93 of [2], Grenander and Szegö introduce a variation of the Toeplitz matrix, $T_{f,N}$, derived from $f(x,p)$ as in (1) where

$$t_{i,j} = f_{j-i}(\frac{1}{2}\ \frac{i+j}{N+1}), \quad 1 \le i,j \le N+1. \tag{2}$$

They also impose the following condition on $f$:

**Condition 1.** *The coefficients $f_k(p)$ are continuous and there exists a constant $M$ such that*

$$\sum_{n=-\infty}^{\infty} max|f_k(p)| \le M. \tag{3}$$

Notice that the matrix $T_{f,N}$ is Hermitian if and only if $f$ is real valued, which we can prove quite easily:

**Theorem 2.1.** *$f$ is real valued $\Leftrightarrow \forall p\ \forall k\ f_k(p) = \overline{f_{-k}(p)}$.*

*Proof.* For the forward direction, consider the formal definition of the Fourier coefficient:

$$\begin{aligned} f_k(p) &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{ikx}\,dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)\overline{e^{-ikx}}\,dx \\ &= \frac{1}{2L}\int_{-L}^{L} \overline{f(x)e^{-ikx}}\,dx \\ &= \overline{f_{-k}(p)}. \end{aligned} \tag{4}$$

For the backwards direction:

$$\begin{aligned} f_k(p) &= \overline{f_{-k}(p)} \\ &= \frac{1}{2L}\int_{-L}^{L} \overline{f(x)e^{-ikx}}\,dx \\ &= \frac{1}{2L}\int_{-L}^{L} \overline{f(x)}e^{ikx}\,dx. \end{aligned} \tag{5}$$

Also we have the relationship that

$$f_k(p) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{ikx}\,dx. \tag{6}$$

$f(x) = \overline{f(x)} \;\Rightarrow f$ is real-valued.

□

Because these matrices are Hermitian (i.e. they fit the conditions of Theorem 2.1), their spectra are composed entirely of real values. This property allows us to, among other things, graph and analyze the distribution of eigenvalues in a way that is easy to visualize. The asymptotic distribution of eigenvalues (as $N \to \infty$) is given by one of the main results of [2].

**Theorem 2.2.** (Szegö Limit Theorem) *Let $f(x,p)$ be a function that satisfies Condition 1. Then $\forall k \geq 1, k \in \mathbb{Z}$:*

$$\lim_{N\to\infty} \frac{1}{N+1} Tr\left((T_{f,N})^k\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 f(x,p)^k\,dp\,dx. \tag{7}$$

Let $\{\lambda_j^{(N)} |\ 0 \leq j \leq N\}$ be the eigenvalues of $T_{f,N}$. Because $T_{f,N}$ is Hermitian, it is not difficult to rewrite (7) as

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{j=0}^{N}\left(\lambda_j^{(N)}\right)^k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 f(x,p)^k\,dp\,dx. \tag{8}$$

Further, it can be shown that this altered form of the Szegö Limit Theorem is just one instance of a more powerful equality:

**Corollary 2.1.** *If $\varphi : \mathbb{R} \to \mathbb{R}$ cont., then*

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{j=0}^{N}\varphi\left(\lambda_j^{(N)}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \varphi(f(x,p))\,dp\,dx. \tag{9}$$

We now introduce a function $\beta$ which, by virtue of (9), will describe the distribution of eigenvalues of $T_{f,N}$ in terms of its symbol, $f(x,p)$. If the $\lambda_j^{(N)}$ are evenly distributed on some fixed interval, $\mathcal{I}$, then one can say that

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{j=0}^{N}\varphi\left(\lambda_j^{(N)}\right) = \int_{\mathcal{I}} \varphi(t)\,dt. \tag{10}$$

Yet, as it becomes apparent in upcoming examples, this is not the case; consequently, we compensate with some correction function, $\beta$, such that

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{j=0}^{N}\varphi\left(\lambda_j^{(N)}\right) = \int_{\mathcal{I}} \varphi(t)\beta(t)\,dt. \tag{11}$$

Combining (9) and (11),

$$\int_{-\pi}^{\pi}\int_{0}^{1} \varphi\left(f(x,p)\right)\, dp\, dx = \int_{\mathcal{I}} \varphi(t)\beta(t)\, dt. \tag{12}$$

Solving for $\beta(t)$ using the equality in (12) requires some ingenuity. First, consider the coarea formula:

**Theorem 2.3.** *Suppose that $\Omega$ is an open set in $\mathbb{R}^n$, and $f$ is a real-valued Lipshitz function on $\Omega$. Then, for an $L^1$ function $\Psi$,*

$$\iint_{\Omega} \Psi(x)|\nabla f(x)|\, dx = \int_{Im(f)} \left( \int_{f^{-1}(x)} \Psi\, ds \right) dx. \tag{13}$$

For our purposes, let

$$\Psi = \frac{\varphi(f(x,p))}{|\nabla f(x,p)|}, \quad \Omega = [-\pi,\pi]\times[0,1] \tag{14}$$

assuming that $\nabla f(x,p)$ does not vanish. Consequently,

$$\begin{aligned}
\int_{-\pi}^{\pi}\int_{0}^{1} \varphi(f(x,p))\, dp\, dx &= \iint_{\Omega} \Psi|\nabla f(x,p)|\, dp\, dx \\
&= \int_{Im(f)} \left( \int_{f^{-1}(t)} \frac{\varphi(f(x,p))}{|\nabla f(x,p)|}\, ds \right) dt \quad \text{(by the coarea fomula)} \\
&= \int_{Im(f)} \left( \int_{f^{-1}(t)} \frac{\varphi(t)}{|\nabla f(x,p)|}\, ds \right) dt \\
&= \int_{Im(f)} \varphi(t) \left( \int_{f^{-1}(t)} \frac{1}{|\nabla f(x,p)|}\, ds \right) dt.
\end{aligned} \tag{15}$$

Therefore

$$\beta(t) = \int_{f^{-1}(t)} \frac{1}{|\nabla f(x,p)|}\, ds. \tag{16}$$

We know of no reasonable method to analytically compute such an integral. Consequently, we focus on approximating $\beta(t)$ instead. We choose the method of orthogonal polynomials, using the normalized *Legendre* polynomials, $P_n$, on the interval $[-1,1]$ where

$$P_n = \frac{p_n}{||p_n||}, \quad p_n = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k}(x+1)^k. \tag{17}$$

Note that because these polynomials are only orthogonal on the interval $[-1,1]$, it is necessary to normalize all functions we consider for this approximation to this interval as well.

Let

$$\beta(x) = \sum_{n=0}^{\infty} a_n P_n(x), \quad a_n = \int_{-1}^{1} \beta(x) P_n(x)\, dx. \tag{18}$$

Since the $p_n$ form a basis [3], the coefficients $a_n$ are guaranteed to exist. Using the equality from (12), we can rewrite $a_n$ as

$$a_n = \int_{-\pi}^{\pi} \int_{0}^{1} P_n(f(x,p))\, dp\, dx. \tag{19}$$

Therefore we can say $\beta \approx \tilde{\beta}$, where

$$\tilde{\beta}(x) = \sum_{n=0}^{M} \left( \int_{-\pi}^{\pi} \int_{0}^{1} P_n(f(x,p))\, dp\, dx \right) P_n(x). \tag{20}$$

**Example 2.1.** Let $f(x,p) = \frac{\cos(x) + p\sin(x)}{\sqrt{2}}$.

The spectrum of $T_{f,N}$ as N varies from 1 to $\infty$ gives us the following graph:

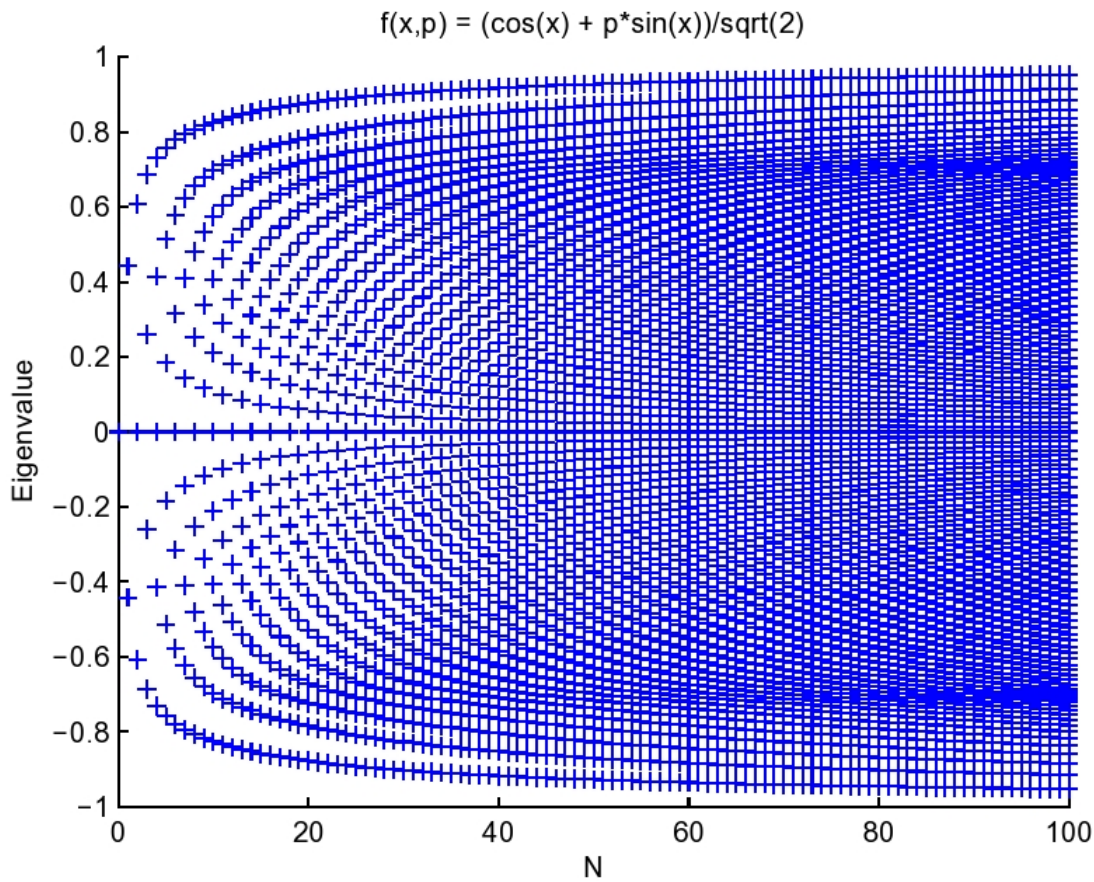



Figure 1: Eigenvalues of $T_{f,N}$ for increasing N

Solving for the distribution function as described above gives us:

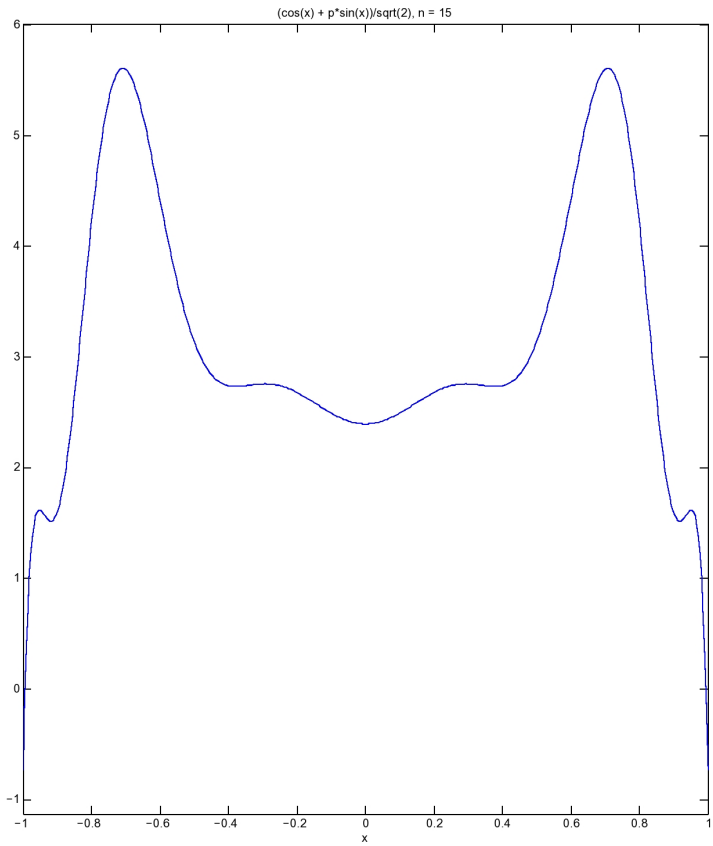



Figure 2: Approximation of the distribution function

Notice that the two peaks of the function in Figure 2 line up exactly with the two areas of highest eigenvalue concentration in Figure 1.

**Example 2.2.** Let $f(x, p) = xp$. As in Example 2.1, we get two graphs:

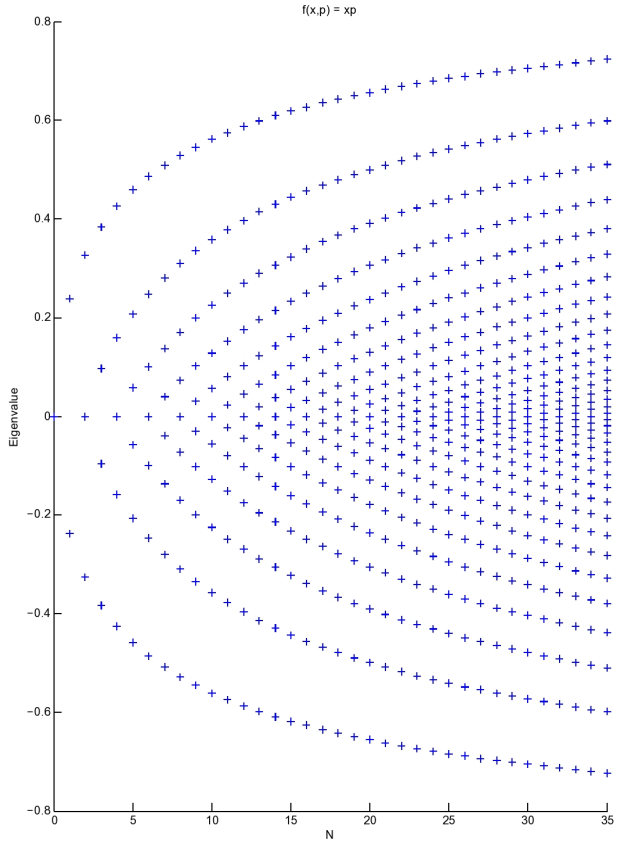



Figure 3: Eigenvalues of $T_{f,N}$ for increasing N

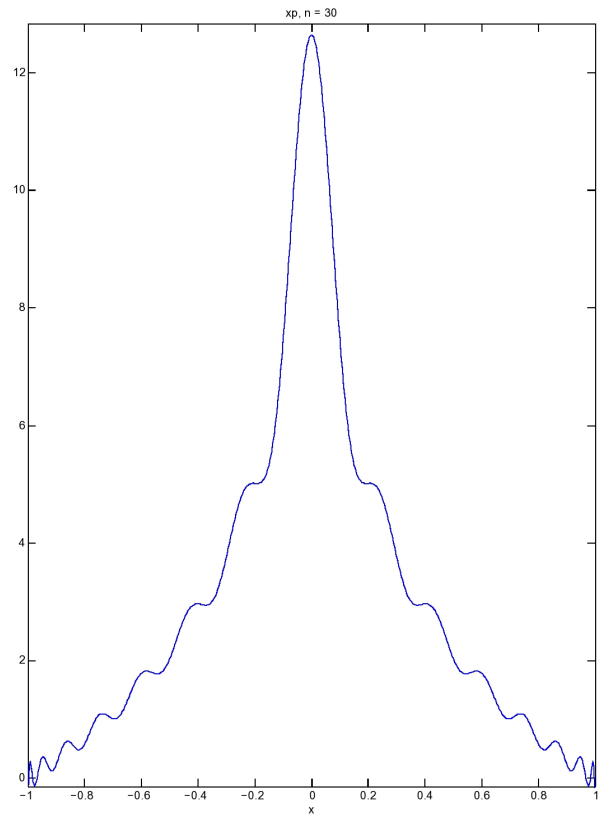



Figure 4: Approximation of the distribution function

As in the previous example, the two graphs agree in regards to the spectral distribution of $f$.

# 3 Berezin-Toeplitz Quantization

While intriguiging, the Weyl quantization method applies solely to symbols that are functions. Consequently, we are motivated to consider a different quantization, the Berezin-Toeplitz, which will allow us to explore symbols that are distributions. Asymptotically, these two quantizations are equivalent when the symbols are smooth (Thm 13.10 in [4]). To define this quantization, we begin working in the context of the Bargmann Space. From now on, the inverse of Planck's constant, $\frac{1}{\hbar}$, will play the role of $N$.

Consider the domain $(x, p) \in [0, 2\pi] \times \mathbb{R}$, and impose a complex variable z s.t. $z = x - ip \in \mathbb{C}$. We can then form a geometric quantization to the Bargmann

Space of the cylinder.

**Definition 3.1.** The Bargmann Space, $\mathcal{B}$, of the cylinder is defined as all $\Psi(z)$ such that

I. $\Psi$ is analytic

II. $\Psi$ is periodic in $z$, i.e. $\Psi(z+2\pi) = \Psi(z)$

III. $||\Psi||^2 = \int_0^{2\pi}\int_{-\infty}^{\infty} |\Psi(z)|^2 e^{-Np^2}\, dp\, dx < \infty, N = \frac{1}{\hbar}$.

The normalization in III. is a consequence of the following Hermitian inner product:

**Definition 3.2.** $\forall\ \Psi_1, \Psi_2 \in \mathcal{B}$,

$$\langle \Psi_1, \Psi_2 \rangle = \int_0^{2\pi}\int_{-\infty}^{\infty} \Psi_1(z)\overline{\Psi_2(z)} e^{-Np^2}\, dp\, dx. \tag{21}$$

Next, to define the Berezin-Toeplitz operator, it is necessary to find an orthonormal basis of $\mathcal{B}$.

**Proposition 3.1.** *$\forall n \in \mathbb{Z}, e^{inz} \in \mathcal{B}$.*

*Proof.* This generalized vector fulfills the three conditions we just set out for $\mathcal{B}$ above. Items I. and II. are obvious. For III., using the inner product defined above:

$$\begin{aligned} ||e^{inz}||^2 &= \int_0^{2\pi}\int_{-\infty}^{\infty} |e^{inz}|^2 e^{-Np^2}\, dp\, dx \\ &= \int_0^{2\pi}\int_{-\infty}^{\infty} |e^{in(x-ip)}|^2 e^{-Np^2}\, dp\, dx \\ &= \int_0^{2\pi}\int_{-\infty}^{\infty} e^{2np-Np^2}\, dp\, dx \\ &= 2\pi e^{\frac{n^2}{N}} \int_{-\infty}^{\infty} e^{-N(p-\frac{n}{N})^2}\, dp. \end{aligned} \tag{22}$$

Let $u = \sqrt{N}(p - \frac{n}{N})$.

$$\begin{aligned} 2\pi e^{\frac{n^2}{N}} \int_{-\infty}^{\infty} e^{-N(p-\frac{n}{N})^2}\, dp &= 2\pi e^{\frac{n^2}{N}} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{N}}\, du \\ &= 2\pi e^{\frac{n^2}{N}} \frac{\sqrt{\pi}}{\sqrt{N}} \\ &= \frac{2\pi^{3/2}}{\sqrt{N}} e^{\frac{n^2}{2N}} < \infty. \end{aligned} \tag{23}$$

□

Since we are looking for an orthonormal basis, we divide by the length of $e^{inz}$, which is the square-root of the value just found above.

**Definition 3.3.** $\forall n \in \mathbb{Z}$,

$$e_n(z) := \frac{e^{inz}}{\sqrt{T}}, \quad T = \frac{2\pi^{3/2}}{\sqrt{N}} e^{\frac{n^2}{2N}}. \tag{24}$$

From this definition, and the way in which we defined the inner product, it follows that $\langle e^{inz}, e^{imz} \rangle = \delta_{nm}$. Consequently we claim that $\{e_n | n \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{B}$. We will not prove that this system is complete, but it is.

Take $f : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$, and we are now able to define the Berezin-Toeplitz operator of $f$.

**Definition 3.4.** $Op(f) : \mathcal{B} \mapsto \mathcal{B}$ is the composition

$$\mathcal{B} \ni \Psi \mapsto f\Psi \mapsto \Pi(f\Psi) \in \mathcal{B} \tag{25}$$

where $\Pi$ is the orthogonal projection

$$\Pi : L^2([0, 2\pi] \times \mathbb{R}, e^{-Np^2}\, dp\, dx) \mapsto \mathcal{B}. \tag{26}$$

In other words, the operator takes an element of $\mathcal{B}$ and multiplies it with $f$, but because $f\Psi \notin \mathcal{B}$ (consider $f(x, p) = \cos(x) + p \sin(x)$), we must then project back onto the Bargmann space, hence the orthogonal projection.

We now have a well-defined Berezin-Toeplitz operator. The next step is to find the matrix, $U_{f,N}$, of $Op(f)$ with respect to $\{e_n, n \in \mathbb{Z}\}$:

$$\begin{aligned} u_{m,n} &= \langle Op(f)(e_n), e_m \rangle \\ &= \langle f e_n, e_m \rangle \quad \text{(because } \Pi \text{ is self-adjoint)} \\ &= \frac{\sqrt{N}}{2\pi^{\frac{3}{2}}} e^{-\frac{n^2+m^2}{2N}} \int_0^{2\pi} \int_{-\infty}^{\infty} f(x,p) e^{in(x-ip)} \overline{e^{im(x-ip)}} e^{-Np^2}\, dp\, dx \\ &= \frac{\sqrt{N}}{2\pi^{\frac{3}{2}}} e^{-\frac{n^2+m^2}{2N}} \int_0^{2\pi} \int_{-\infty}^{\infty} f(x,p) e^{i(n-m)x} e^{(n+m)p - Np^2}\, dp\, dx. \end{aligned} \tag{27}$$

As an example, we compute the case of $Op(f)$ where $f = p$.

**Example 3.1.**

$$\begin{aligned} Op(p) &= \frac{\sqrt{N}}{2\pi^{\frac{3}{2}}} e^{-\frac{n^2+m^2}{2N}} \int_0^{2\pi} \int_{-\infty}^{\infty} p e^{i(n-m)x} e^{(n+m)p - Np^2}\, dp\, dx \\ &= \delta_{m,n} \sqrt{\frac{N}{\pi}} e^{\frac{-n^2}{N}} \int_{-\infty}^{\infty} p e^{2np - Np^2}\, dp \\ &= \delta_{m,n} \sqrt{\frac{N}{\pi}} e^{\frac{-n^2}{N}} \int_{-\infty}^{\infty} p e^{N(p - \frac{n}{N})^2 - \frac{n^2}{N}}\, dp. \end{aligned} \tag{28}$$

Let $q = p - \frac{n}{N}$.

$$\begin{aligned} Op(p) &= \delta_{m,n}\sqrt{\frac{N}{\pi}}e^{\frac{-n^2}{N}}\int_{-\infty}^{\infty}(q+\frac{n}{N})e^{-Nq^2+\frac{n^2}{N}}\,dq \\ &= \delta_{m,n}\sqrt{\frac{N}{\pi}}\int_{-\infty}^{\infty}(q+\frac{n}{N})e^{-Nq^2}\,dq \\ &= \delta_{m,n}\sqrt{\frac{N}{\pi}}\left(\int_{-\infty}^{\infty}qe^{-Nq^2}\,dq+\frac{n}{N}\int_{-\infty}^{\infty}e^{-Nq^2}\,dq\right) \\ &= \delta_{m,n}\sqrt{\frac{N}{\pi}}\frac{n}{N}\sqrt{\frac{\pi}{N}} \\ &= \delta_{m,n}\frac{n}{N}. \end{aligned} \tag{29}$$

This proves the following lemma:

**Lemma 3.2.** *The spectrum of $Op(f)$ where $f = p$ consists of the eigenvalues $\frac{n}{N}, n \in \mathbb{Z}$. And the $e_n$ are the associated eigenfunctions.*

We will now truncate $Op(f)$ as follows: Let $\mathcal{H}$ be the span of eigenfunctions of $Op(p)$ with eigenvalues in $[0,1]$. Consider that

$$0 \le n \le N \iff 0 \le \frac{n}{N} \le 1. \tag{30}$$

Therefore $\mathcal{H} = span\{e_n, n = 0, 1, \ldots, N\}$. By the quantum-classical correspondence, $\mathcal{H}$ corresponds to the part of the cylinder defined by $0 \le p \le 1$. Out of the infinite matrix $(u_{m,n})_{m,n\in\mathbb{Z}}$ of $Op(f)$, we extract the block corresponding to $\mathcal{H}$, namely $(u_{m,n})_{0\le m,n\le N}$. In fact, for ease of computation, we replace it with the matrix $V = (v_{m,n})_{0\le m,n\le N}$ where

$$v_{m,n} = \frac{\sqrt{N}}{2\pi^{\frac{3}{2}}}e^{-\frac{n^2+m^2}{2N}}\int_0^{2\pi}\int_0^1 f(x,p)e^{i(n-m)x}e^{(n+m)p-Np^2}\,dp\,dx. \tag{31}$$

That is, where the integration is over the part of the cylinder corresponding to $\mathcal{H}$. This is justified by the following lemma:

**Lemma 3.3.** *If $1 \le m, n \le N-1$, then $|u_{m,n} - v_{m,n}|$ is exponentially small as $N \to \infty$.*

*Proof.* It is enough to show that

$$e^{-\frac{n^2+m^2}{2N}}\left(\int_1^{\infty}e^{(n+m)p-Np^2}\,dp+\int_{-\infty}^{0}e^{(n+m)p-Np^2}\,dp\right) \tag{32}$$

is exponentially small as $N \to \infty$.

*Case* 1. First, consider the integral on the left:

$$e^{-\frac{n^2+m^2}{2N}}\int_1^{\infty}e^{(n+m)p-Np^2}\,dp = e^{-\frac{n^2+m^2}{2N}}\int_1^{\infty}e^{N(p-\frac{n+m}{2N})^2-\frac{(n+m)^2}{4N^2}} \tag{33}$$

Let $s = \sqrt{N}\left(p - \frac{n+m}{2N}\right)$.

$$\begin{aligned} e^{-\frac{n^2+m^2}{2N}} \int_1^\infty e^{N(p-\frac{n+m}{2N})^2 - \frac{(n+m)^2}{4N^2}} &= \frac{e^{-\frac{(n-m)^2}{4N}}}{\sqrt{N}} \int_{\sqrt{N}-\frac{n+m}{2\sqrt{N}}}^\infty e^{-s^2}\, ds \\ &= \frac{e^{-\frac{(n-m)^2}{4N}}}{\sqrt{N}} \mathrm{erfc}(\sqrt{N} - \frac{n+m}{2\sqrt{N}}) \end{aligned} \tag{34}$$

where $\mathrm{erfc}(x)$ is the complemtary error function, which is exponentially small as $N \to \infty$.

*Case* 2. For the second integral of (32), a similar result is obtained with the simplification process from Case 1. We end up with the following expression:

$$\frac{e^{-\frac{(n-m)^2}{4N}}}{\sqrt{N}} \int_{-\infty}^{-\frac{n+m}{2\sqrt{N}}} e^{-s^2}\, ds. \tag{35}$$

Since $e^{-s^2}$ is an even function, this expression also reduces to some constant multiplied by $\mathrm{erfc}(x)$.

□

Since the cases where $n = N = m$ and $n = 0 = m$ are such a small percentage of all the possible combinations of $0 \leq n, m \leq N$ for large N, it is reasonable to neglect the small error that they cause when changing the bounds of $p$ as in (31). For simplicity, let

$$v_{m,n} = A^N_{m,n} \int_0^{2\pi} \int_0^1 f(x,p) e^{i(n-m)x} e^{(n+m)p - Np^2}\, dp\, dx \tag{36}$$

where

$$A^N_{m,n} = \frac{\sqrt{N}}{2\pi^{\frac{3}{2}}} e^{-\frac{n^2+m^2}{2N}}. \tag{37}$$

At last, (36) is our Berezin-Toeplitz quantization on the cut cylinder $0 \leq p \leq 1$. It is interesting to compare the Weyl and Berezin-Toeplitz quantizations together for the same function, as asymptotically they should be the same. For comparision, it is necessary to normalize the Berezin-Toeplitz operator by $\frac{1}{\sqrt{N}}$; otherwise, the eigenvalues of $V$ will tend to infinity.

**Example 3.2.** Let $f(x,p) = p\cos(x)$.

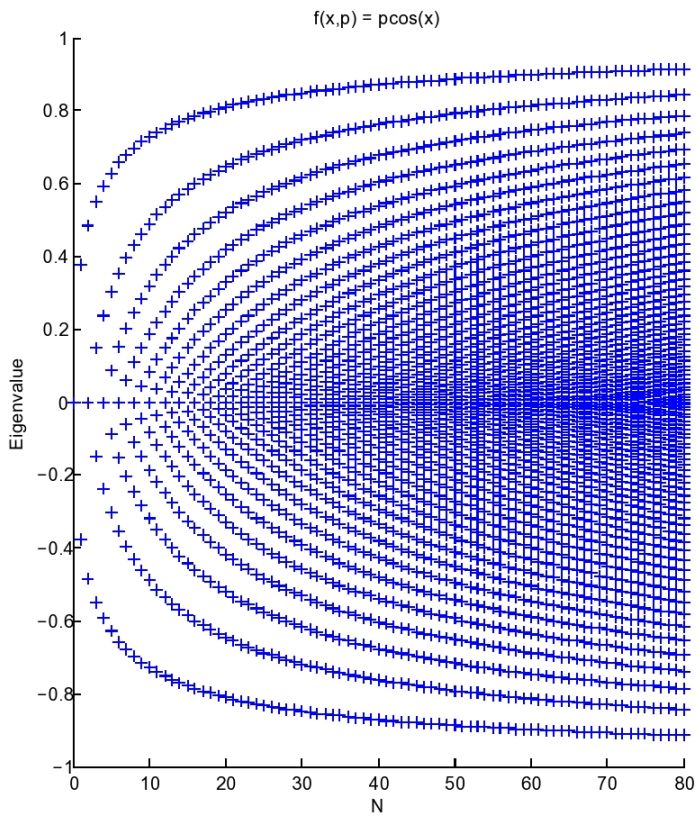



Figure 5: Weyl quantization

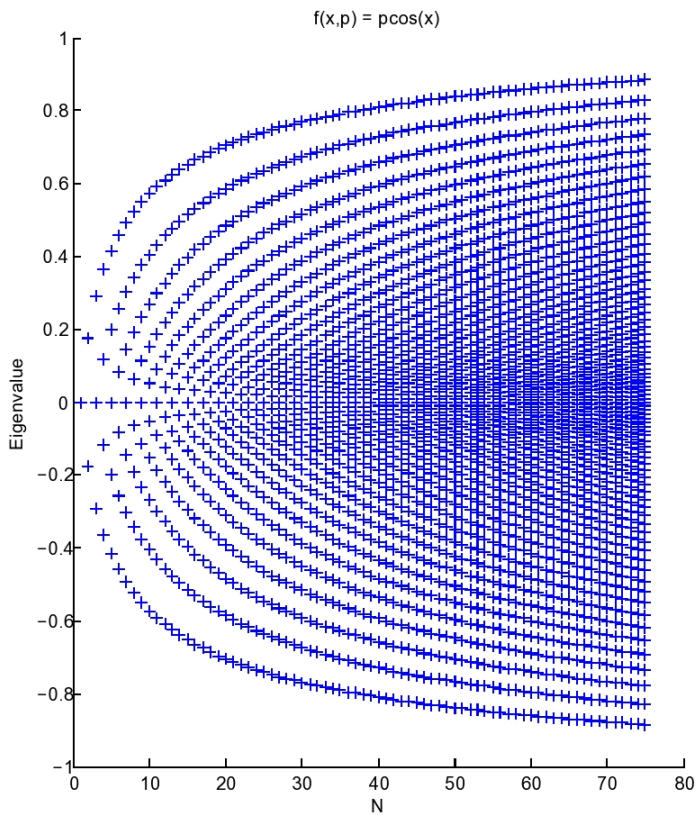



Figure 6: Berezin-Toeplitz quantization

Equation (36) immediately extends to the case where $f$ is a generalized function.

# 4 Examples of B-T Operators with Singular Symbols

We now consider two examples in which the symbol is a distribution. These examples will motivate our main result.

**Example 4.1.** Let $f(x,p) = \delta(x_0, p_0)$, the standard Dirac delta function. Substituting into the Berezin-Toeplitz quantization,

$$\begin{aligned} v_{m,n} &= A^N_{m,n} \int_0^{2\pi} \int_0^1 \delta(x_0, p_0) e^{i(n-m)x} e^{(n+m)p - Np^2} \, dp \, dx \\ &= A^N_{m,n} e^{i(n-m)x_0} e^{(n+m)p_0 - Np_0^2}. \end{aligned} \tag{38}$$

For $x_0 = 1, p_0 = 1/3$:

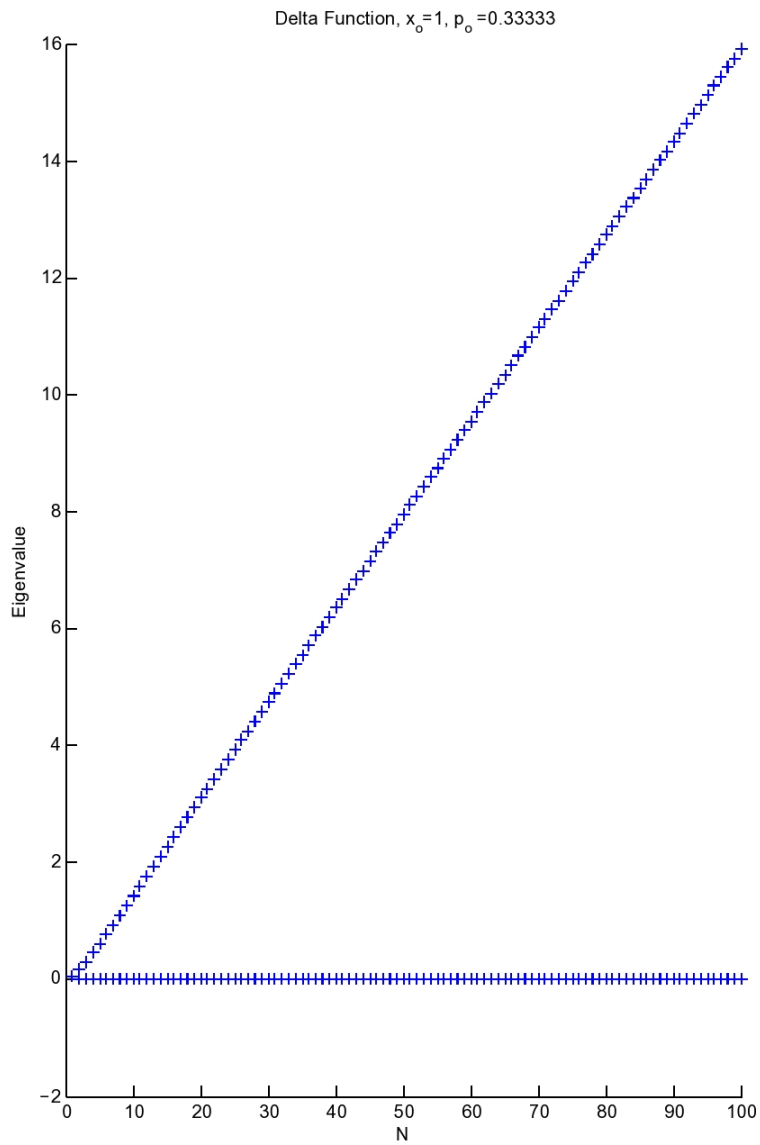



Figure 7: Spectrum of the Dirac delta function

The Dirac delta function appears to be a linear projector multiplied by a linearly increasing constant in the context of this quantization.

While the above example is interesting, there is very little to describe about the symbol's distibution of eigenvalues. Consequently, we are motivated to consider a more complicated symbol–the Dirac delta function associated with a function, $\delta_\gamma(x,p)$–which we define as follows:

**Definition 4.1.** $\forall\ \varphi : [0, 2\pi] \times [0, 1] \to \mathbb{R}$,

$$\int_0^{2\pi} \int_0^1 \delta_\gamma(x,p)\varphi(x,p)\, dp\, dx = \int_\gamma \varphi\, ds. \tag{39}$$

Applying the Berezin-Toeplitz quantization,

$$\begin{aligned} v_{m,n} &= A^N_{m,n} \int_0^{2\pi} \int_0^1 \delta_\gamma(x,p) e^{i(n-m)x} e^{(n+m)p - Np^2}\, dp\, dx \\ &= A^N_{m,n} \int_\gamma e^{i(n-m)x} e^{(n+m)p - Np^2}\, ds. \end{aligned} \tag{40}$$

**Example 4.2.** Consider the ellipse

$$\left(\frac{x}{2\pi}\right)^2 + p^2 = 1, \quad 0 \le x \le \frac{\pi}{2}, \quad 0 \le p \le 1. \tag{41}$$

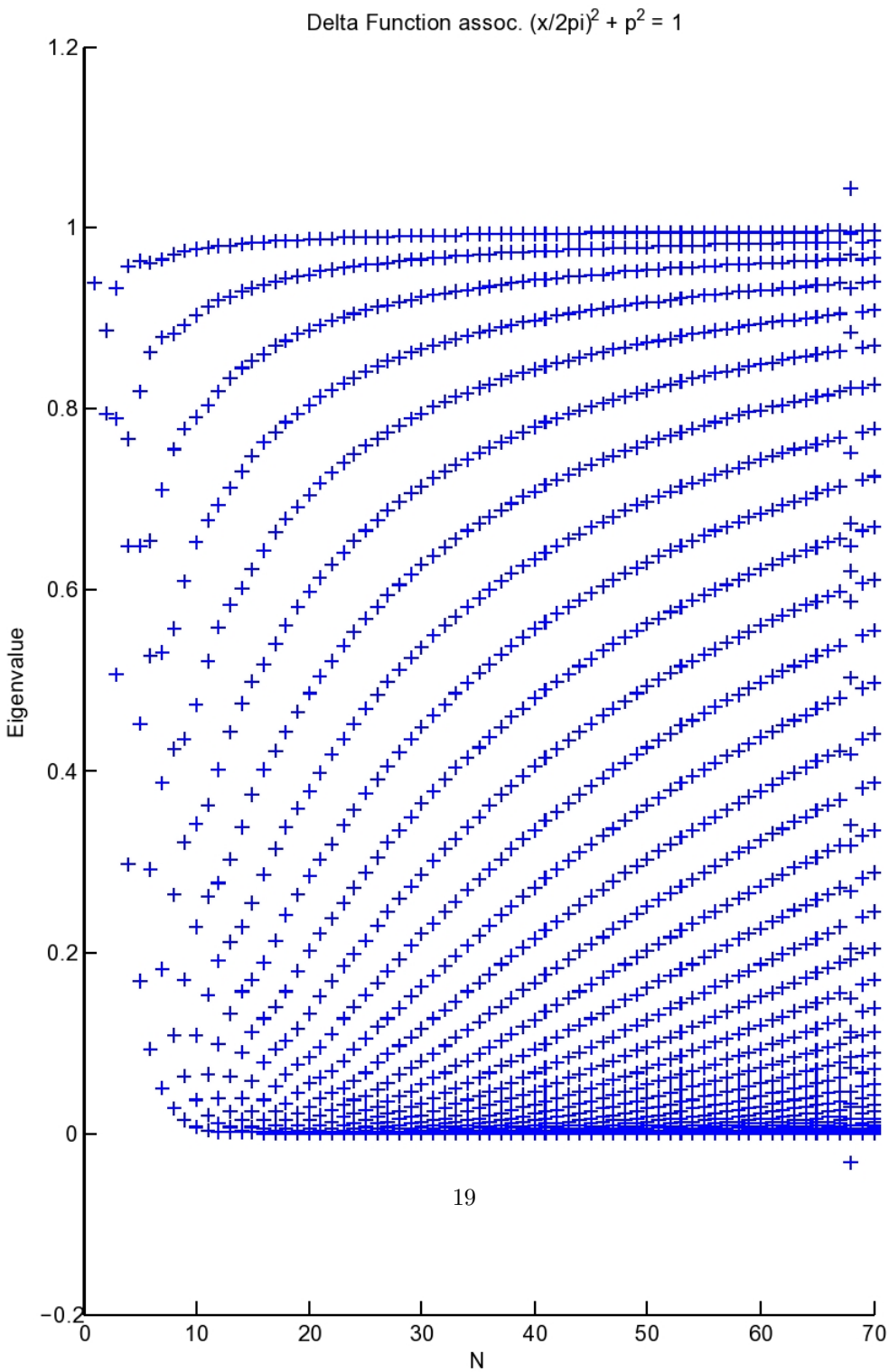
We apply the parametrization

$$x = 2\pi\cos(t), \quad p = \sin(t), \quad ds = \sqrt{\frac{dx}{dt}^2 + \frac{dp}{dt}^2}\, dt, \quad 0 \le t \le \frac{\pi}{2}, \tag{42}$$

to get the following result:

$$v_{m,n} = A^N_{m,n} \int_0^{\frac{\pi}{2}} e^{i(n-m)2\pi\cos(t)} e^{(n+m)\sin(t) - N\sin^2(t)} \sqrt{(-2\pi\sin(t))^2 + \cos^2(t)}\, dt.$$

Using MATLAB to graph the eigenvalues of these matrices for varying N, it is necessary to use numerical integration, specifically adapative Simpson quadrature in this instance, to get compile times under 8 hours. When computing the matrix for some values of N, a singularity occurs in the quadrature process causing a skewed approximation to occur. The overall trend of the graph, however, is still an accurate approximation:


Delta Function assoc. (x/2pi)^2 + p^2 = 1
Eigenvalue
N
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
10
20
30
40
50
60
70

# 5 Main Results

In this section we consider a case where the symbol is a Dirac delta function associated with a curve. In this case the spectrum can be computed exactly, and we obtain a Szegö Limit Theorem. We then make a conjecture that generalizes this theorem.

Let $f(x,p) = \delta_\gamma(x,p)$, where $\gamma$ is the parametrized curve

$$\begin{cases} x = t \ , \ 0 \leq t \leq 2\pi \\ p = p \ , \ 0 \leq p \leq 1 \end{cases} \tag{43}$$

Applying the Berezin-Toeplitz quantization, we find the matrix elements

$$\begin{aligned} u_{m,n} &= A^N_{m,n} \int_0^{2\pi} \int_0^1 \delta_\gamma(x,p) e^{i(n-m)x} e^{(n+m)p - Np^2} \, dp \, dx \\ &= A^N_{m,n} e^{(n+m)p - Np^2} \int_0^{2\pi} e^{i(n-m)t} \, dt \\ &= 2\pi A^N_{m,n} \delta_{mn} e^{(n+m)p - Np^2}, \end{aligned} \tag{44}$$

where $\delta_{mn}$ is the Kroenecker delta. This simplifies to

$$u_{m,n} = \delta_{mn} \sqrt{\frac{N}{\pi}} e^{-(\sqrt{N}p - \frac{n}{\sqrt{N}})^2}. \tag{45}$$

Since the diagonal entries of a diagonal matrix are its eigenvalues, after normalizing:

$$\lambda_k^{(N)} = e^{-(\sqrt{N}p - \frac{k}{\sqrt{N}})^2}. \tag{46}$$

In Section 2, we stated the Szegö Limit Theorem. We now state and prove a similar two-term Szegö limit expansion for the class of delta functions defined by (43).

**Theorem 5.1.** *$\forall \mathcal{W} \in C^2$, as $N \to \infty$*

$$\frac{1}{N} \sum_{k=1}^N \mathcal{W}(\lambda_k^{(N)}) = \mathcal{W}(0) + \frac{1}{2\sqrt{N}} \int_0^1 \frac{\mathcal{W}(s) - \mathcal{W}(0)}{s\sqrt{-\log(s)}} \, ds + O\left(Ne^{-Np^2}\right). \tag{47}$$

*Proof.* We begin with

$$\frac{1}{N} \sum_{k=1}^N \mathcal{W}(\lambda_k^{(N)}) \tag{48}$$

and then replace $\lambda_k^{(N)}$ with the expression from (46):

$$\frac{1}{N}\sum_{k=1}^{N}\mathcal{W}(e^{-(\sqrt{N}p-\frac{k}{\sqrt{N}})^2}). \tag{49}$$

We then split $\frac{1}{N}$ into two equal factors:

$$\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\mathcal{W}(e^{-(\sqrt{N}p-\frac{k}{\sqrt{N}})^2})\frac{1}{\sqrt{N}}. \tag{50}$$

This is allows us to consider (48) as a Riemann sum with step $\frac{1}{\sqrt{N}}$ of the following integral:

$$\frac{1}{\sqrt{N}}\int_0^{\sqrt{N}}\mathcal{W}(e^{-(\sqrt{N}p-u)^2})\,du. \tag{51}$$

We will use the following well-known lemma to calculate an exact error bound on this Riemann approximation.

**Lemma 5.2.** *Let $f(x)$ be a bounded function on a bounded interval $[a,b]$, partitioned into $p$ subintervals $[x_{k-1},x_k], k=1,\ldots,p$. In each subinterval $[x_{k-1},x_k]$ choose $x_k^*, x_{k-1}\le x_k^*\le x_k$. Then*

$$\left|\sum_{k=1}^{p}(x_k-x_{k-1})f(x_k^*)-\int_a^b f(x)\,dx\right|\le\frac{b-a}{2}\max_{[a,b]}|f'(x)|\delta_{max} \tag{52}$$

*where $\delta_{max}=\max_k(x_k-x_{k-1})$.*

For our case:

$$b-a=\sqrt{N},\quad \delta_{max}=\frac{1}{\sqrt{N}},\quad f=\mathcal{W}(e^{-(\sqrt{N}p-u)^2}) \tag{53}$$

and

$$\max_{[a,b]}|f'(x)|=f'(0)=2Np^3e^{-Np^2}\mathcal{W}'(e^{-Np^2}) \tag{54}$$

where

$$f'(x)=\mathcal{W}'(e^{-(\sqrt{N}p-\frac{x}{\sqrt{N}})^2})\frac{2}{\sqrt{N}}(\sqrt{N}p-\frac{x}{\sqrt{N}})^3e^{-(\sqrt{N}p-\frac{x}{\sqrt{N}})^2} \tag{55}$$

Consequently,

$$\left|\frac{1}{N}\sum_{k=1}^{N}\mathcal{W}(\lambda_k^{(N)})-\frac{1}{\sqrt{N}}\int_0^{\sqrt{N}}\mathcal{W}(e^{-(\sqrt{N}p-u)^2})\,du\right|\le 2Np^3e^{-Np^2}\mathcal{W}'(e^{-Np^2}). \tag{56}$$

At this point we have the relationship

$$\frac{1}{N}\sum_{k=1}^{N}\mathcal{W}(\lambda_k^{(N)}) = \frac{1}{\sqrt{N}}\int_0^{\sqrt{N}}\mathcal{W}(e^{-(\sqrt{N}p-u)^2})\,du + O\left(Ne^{-Np^2}\right) \tag{57}$$

We now analyze asymptotically the integral in (57). For simplicity, apply the change of variable $t = \frac{u-\sqrt{N}p}{\sqrt{N}}$:

$$\int_{-p}^{1-p}\mathcal{W}(e^{-Nt^2})\,dt. \tag{58}$$

**Lemma 5.3.**

$$\int_{-p}^{1-p}\mathcal{W}(e^{-Nt^2})\,dt = \mathcal{W}(0) + \frac{1}{\sqrt{N}}\int_0^1\frac{\mathcal{W}(s)-\mathcal{W}(0)}{s\sqrt{-\log(s)}}\,ds + O(e^{-Np^2}). \tag{59}$$

*Proof.* Split up the left hand side of (59) as

$$\int_0^p\mathcal{W}(e^{-Nt^2})\,dt + \int_0^{1-p}\mathcal{W}(e^{-Nt^2})\,dt. \tag{60}$$

Note that the bounds on the first integral are correct because $\mathcal{W}(e^{-Nt^2})$ is guaranteed to be an even function. Now apply the change of variable $s = e^{-Nt^2}$ to obtain

$$\frac{1}{2\sqrt{N}}\left[\int_{e^{-Np^2}}^1\frac{\mathcal{W}(s)}{s\sqrt{-\log s}}\,ds + \int_{e^{-N(1-p)^2}}^1\frac{\mathcal{W}(s)}{s\sqrt{-\log s}}\,ds\right]. \tag{61}$$

Let

$$\mathcal{I}_N = \frac{1}{2\sqrt{N}}\int_{e^{-Np^2}}^1\frac{\mathcal{W}(s)}{s\sqrt{-\log(s)}}\,ds, \quad \mathcal{J}_N = \frac{1}{2\sqrt{N}}\int_{e^{-N(1-p)^2}}^1\frac{\mathcal{W}(s)}{s\sqrt{-\log(s)}}\,ds \tag{62}$$

and

$$\frac{\mathcal{W}(s)}{s} = \frac{\mathcal{W}(0)}{s} + \mathcal{Z}(s), \quad \text{where } \mathcal{Z}(s) = \frac{\mathcal{W}(s)-\mathcal{W}(0)}{s}. \tag{63}$$

The following analysis is performed for $\mathcal{I}_N$ only, but is identical to that of $\mathcal{J}_N$. Substituting into $\mathcal{I}_N$ we get

$$\mathcal{I}_N = \mathcal{W}(0)\left(\frac{1}{2\sqrt{N}}\int_{e^{-Np^2}}^1\frac{1}{s\sqrt{-\log(s)}}\,ds\right) + \frac{1}{2\sqrt{N}}\int_{e^{-Np^2}}^1\frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\,ds. \tag{64}$$

Consider the first integral expression in equation (64). By reversing our most recent change of variables, it reduces to just $p$, and our equation simplifies to

$$\mathcal{I}_N = \mathcal{W}(0)p + \frac{1}{2\sqrt{N}} \int_{e^{-Np^2}}^{1} \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\, ds. \tag{65}$$

Now consider the second integral in (64). We need to show that this integral can be rewritten as

$$\int_0^1 \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\, ds. \tag{66}$$

Or, in other words,

$$\int_{e^{-Np^2}}^{1} \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\, ds - \int_0^1 \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\, ds = \int_0^{e^{-Np^2}} \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\, ds \tag{67}$$

and that this difference is exponentially small. Note that the integrand is a continuous function on the domain of integration, for $N > 0$. Therefore we can apply the Mean Value Theorem for integrals: $\forall N\ \exists s_N \in [0, e^{-Np^2}]$ such that

$$\int_0^{e^{-Np^2}} \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\, ds = e^{-Np^2} \frac{\mathcal{Z}(s_N)}{\sqrt{-\log(s_N)}}. \tag{68}$$

If we take, for example

$$C = \sup_{s \in [0, \frac{1}{2}]} \left| \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}} \right| \tag{69}$$

then we have that for all sufficiently large N

$$\left| \lim_{N \to \infty} \int_{e^{-Np^2}}^{1} \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\, ds - \int_0^1 \frac{\mathcal{Z}(s)}{\sqrt{-\log(s)}}\, ds \right| \leq Ce^{-Np^2}. \tag{70}$$

Our final result is that

$$\mathcal{I}_N = \mathcal{W}(0)p + \frac{1}{2\sqrt{N}} \int_0^1 \frac{\mathcal{W}(s) - \mathcal{W}(0)}{s\sqrt{-\log(s)}}\, ds + O(e^{-Np^2}). \tag{71}$$

Similarly,

$$\mathcal{J}_N = \mathcal{W}(0)(1-p) + \frac{1}{2\sqrt{N}} \int_0^1 \frac{\mathcal{W}(s) - \mathcal{W}(0)}{s\sqrt{-\log(s)}}\, ds + O(e^{-Np^2}). \tag{72}$$

Summing the two together we obtain

$$\mathcal{I}_N + \mathcal{J}_N = \mathcal{W}(0) + \frac{1}{\sqrt{N}} \int_0^1 \frac{\mathcal{W}(s) - \mathcal{W}(0)}{s\sqrt{-\log(s)}}\, ds + O(e^{-Np^2}). \tag{73}$$

This proves the lemma. □

We can now use the triangle inequality to finish the proof of the theorem. Let

$$\mathcal{P} = \frac{1}{N}\sum_{k=1}^{N}\mathcal{W}(\lambda_k^{(N)}) \quad \mathcal{Q} = \mathcal{W}(0)+\frac{1}{2\sqrt{N}}\int_0^1 \frac{\mathcal{W}(s)-\mathcal{W}(0)}{s\sqrt{-\log(s)}}\,ds \quad \mathcal{R} = \int_{-p}^{1-p}\mathcal{W}(e^{-Nt^2})\,dt \tag{74}$$

By the triangle inequality,

$$|\mathcal{P}-\mathcal{Q}| \leq |\mathcal{P}-\mathcal{R}| + |\mathcal{R}-\mathcal{Q}| \tag{75}$$

By (56) and Lemma 5.3,

$$\begin{aligned} |\mathcal{P}-\mathcal{R}| &\leq O\left(Ne^{-Np^2}\right) + O(e^{-Np^2}) \\ &\leq O\left(Ne^{-Np^2}\right). \end{aligned} \tag{76}$$

This proves the theorem.

□

Our theorem implies that the spectrum for any delta function as defined by equation (43) will have the following distribution function away from $s = 0$:

$$\beta(s) = \frac{1}{s\sqrt{-\log(s)}} \tag{77}$$

We can visualize this with an example.

**Example 5.1.** Let $f(x,p)$ be the Dirac delta function associated with $x = t,\ p = \frac{1}{2}$.

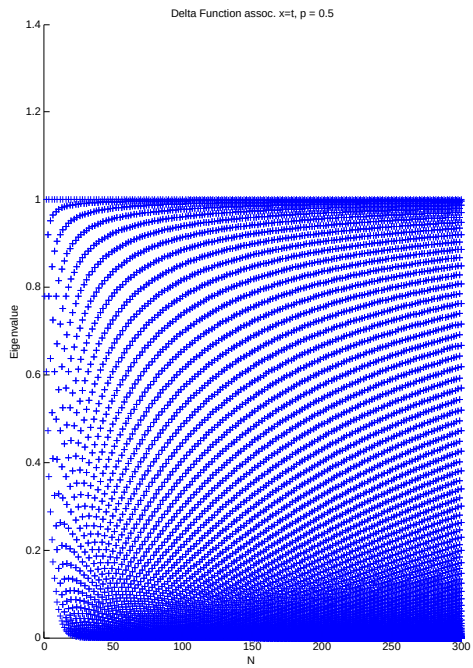



Figure 9: Spectrum for increasing $N$

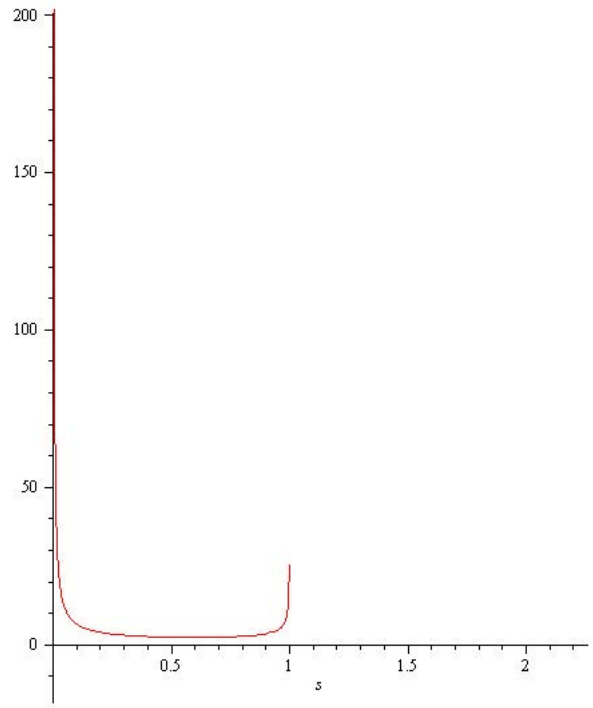



Figure 10: $\beta(s) = \frac{1}{s\sqrt{-\log(s)}}$

After proving this theorem and studying various related examples, we are motivated to make the following conjecture:

**Conjecture 5.1.** For all Dirac delta functions associated with a curve and a density on it, there exists a two-term Szegö limit expansion, with a leading term $\mathcal{W}(0)$ and a second term of order $O(\frac{1}{\sqrt{N}})$ which behaves as in the following examples.

**Example 5.2.** Let $f(x,p)$ be the delta function associated with $x = 2\pi t, p = t$.

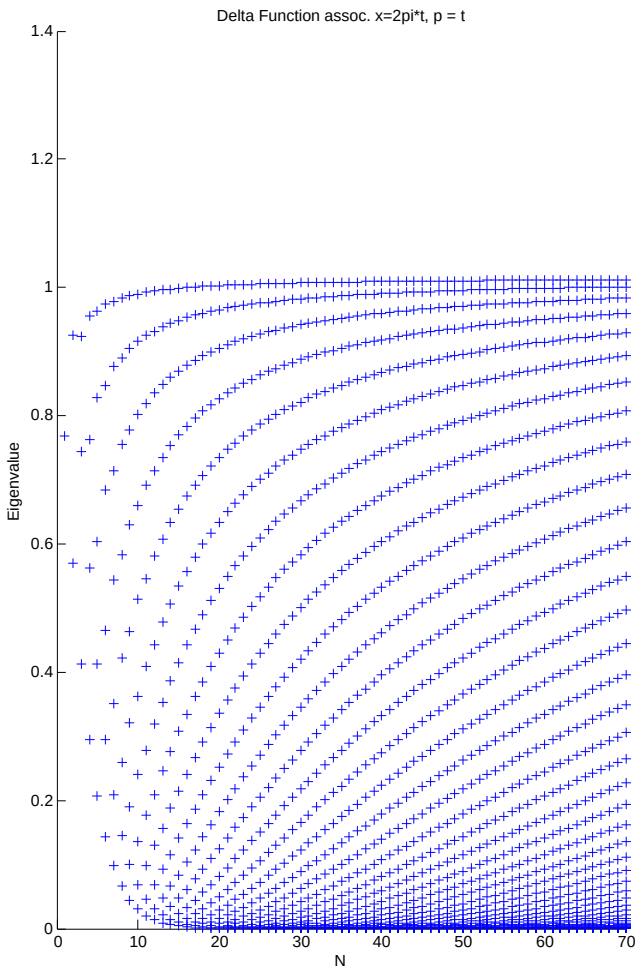



Figure 11: Spectrum for increasing $N$

**Example 5.3.** Let $f(x, p)$ be the delta function associated with $x = 2\pi t^2, p = t$.

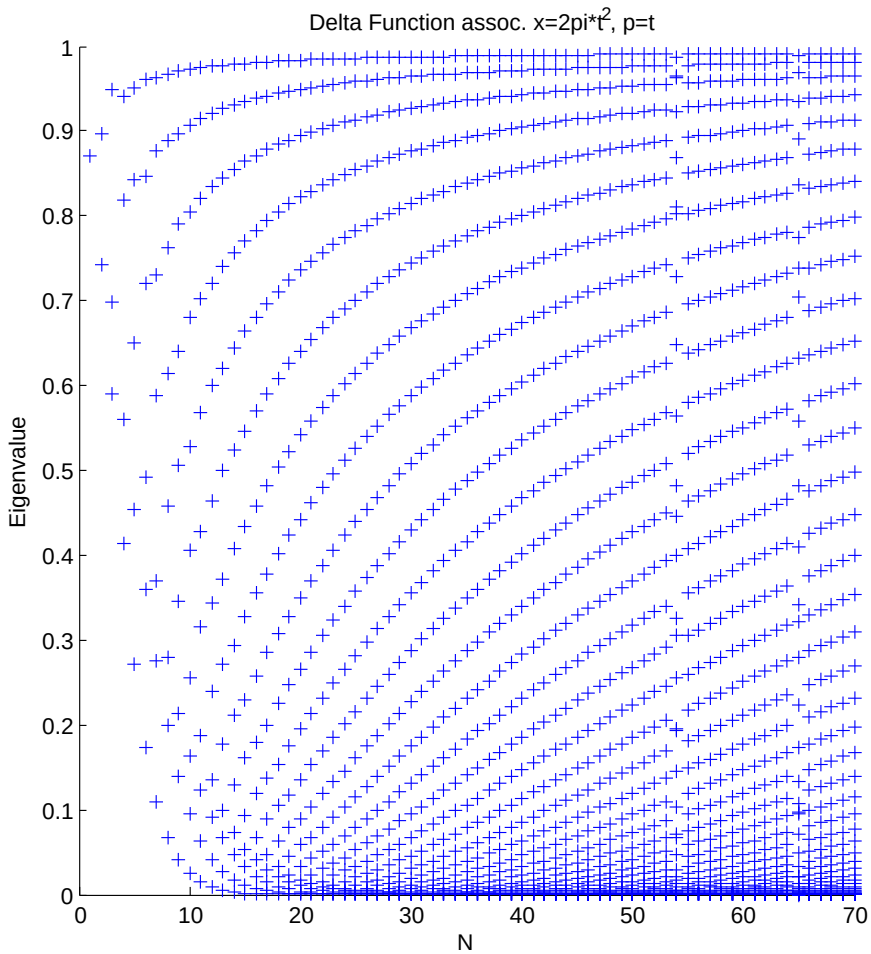



Figure 12: Spectrum for increasing $N$

# 6 Selected MATLAB Code

## Figure 1

```
%initialize and create variables
E = [];

for n = 0:100

    %initialize toeplitz matrix
    toep = zeros(n+1);

    %based on condition, put correct value
    for i = 1:(n+1)
        for j = 1:(n+1)
            if i-j == 1
                toep(i,j) = f1(.5*(i+j)/(n+1))*(1/sqrt(2));
            end
            if i - j == -1
                toep(i,j) = fneg1(.5*(i+j)/(n+1))*(1/sqrt(2));
            end
        end
    end

     %obtain a column vector of eigenvalues of toeplitz matrix
     EIG = eig(toep);

     %obtain length of eigenvalue vector
     len = length(EIG(:,1));

     %create a temporary matrix concatenating a vector of appropriate length
     %of zeroes to the eigenvalue vector
     temp = [zeros(len, 1) EIG];

     %put appropriate value into the left hand column
     for k = 1:len
         temp(k) = n;
     end

     %finally, concatenate temp to our E matrix which is what we will
     %ultimately be plotting
     E = [E; temp];
end

%scatter plot of E
figure;
```

```
scatter(E(:,1), E(:,2), '+')
title('f(x,p) = (cos(x) + p*sin(x))/sqrt(2)')
xlabel('N')
ylabel('Eigenvalue')
```

**Note: this code calls the two functions below**

```
function [y] = f1(x)
    y = .5 + x/(2*1i);

function [y] = fneg1(x)
    y = .5 - x/(2*1i);
```

## Figure 2

```
%initialize and create variables
syms x p;
apprx = 0;
N = 15;

    for n = 0:N

        %polynomial basis vector, [x^n, n^n-1, ...,x^0]
        E = [];
        for m = 1:(n+1)
            temp = x^((n+1)-m);
            E = [E; temp];
        end

        %Jacobi polynomial of length for n-1
        P_n = orth_poly('Jacobi',n,0,0) * E;


        %normalize the Jacobi polynomial
        P_norm = sqrt(int((P_n)^2,x,-1,1));
        P_n = P_n/P_norm;


        p_n = P_n;

        %Find p_n(f(x,p))
        G = subs(P_n,(cos(x) + (p)*sin(x))/(sqrt(2)),x);

        %Find a_n by double integrating p_n(f(x,p))
        a_n = int(int(G,x,0,2*pi),p,0,1);

        %iterate into our jacobi polynomial approximation
        apprx = apprx + a_n*p_n;
```

```
    end

    %Plot the Jacobi polyn approximation
    figure;
    ezplot(apprx, -1,1)
    title(['(cos(x) + p*sin(x))/sqrt(2), n = ', num2str(N)])
```

**Figure 8**

```
%initialize and create variables
E = [];
syms x t

h = waitbar(0,'Please wait...');
N = 70;
for n = 1:N
    %initialize toeplitz matrix to zeros
    toep = zeros(n);
    norm = sqrt(n/pi);
    un_norm = 1;

    %based on condition, put correct value
    for i = 1:n
        for j = 1:n
            A_ijn = (sqrt(n)/(2*(pi)^(3/2)))*exp(-1*(i^2 + j^2)/(2*n));

            %exp((1i*(j-i)*2*pi*cos(t)) + ((i+j)*sin(t)) - (n*((sin(t))^2)))*sqr
            f = makefun_ellipse(i,j,n);
            q = matlabFunction(f(t));


            toep(i,j) = A_ijn*quadgk(q,0,pi/2)/norm;
        end
    end
% end
% toep

    %obtain a column vector of eigenvalues of toeplitz matrix
    EIG = eig(toep);

    %obtain length of eigenvalue vector
    len = length(EIG(:,1));

    %create a temporary matrix concatenating a vector of appropriate length
    %of zeroes to the eigenvalue vector
    temp = [zeros(len, 1) EIG];
```

```
    %put appropriate value into the left hand column
    for k = 1:len
        temp(k) = n;
    end

    %finally, concatenate temp to our E matrix which is what we will
    %ultimately be plotting
    E = [E; temp];

    waitbar(n/N)
end
close(h)

%scatter plot of E
figure;
scatter(E(:,1), E(:,2), '+')
title('Delta Function assoc. (x/2pi)^2 + p^2 = 1')
xlabel('N')
ylabel('Eigenvalue')
```

**Note: this code calls the following function**

```
function fcn = makefun_ellipse(i,j,n)

fcn = @parabola;

    function y = parabola(t)
        y = exp((1i*(j-i)*2*pi*cos(t)) + ((i+j)*sin(t)) - (n*((sin(t))^2)))*sqrt
    end
end
```

# References


[1] Böttcher, Albrecht and Bernd Silbermann. *Introduction to Large Truncated Toeplitz Matrices*. New York: Springer, 1999. Print.

[2] Grenander, Ulf, and Gabor Szegö. *Toeplitz Forms and Their Applications*. 2nd ed. New York: Chelsea, 1984. Print. 81-96

[3] Jackson, Dunham. *Fourier Series and Orthogonal Polynomials* Vol. 6. Menasha: Mathematical Association of America, 1941. Print. The Carus Mathematical Monographs.

[4] Zworski, Maciej. *Semiclassical Analysis*. Boston: AMS. In Press.